LIMIT-PERIODIC SCHRÖDINGER OPERATORS IN THE REGIME OF POSITIVE LYAPUNOV EXPONENTS

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Abstract. We investigate the spectral properties of discrete one-dimensional Schrödinger operators whose potentials are generated by continuous sampling along the orbits of a minimal translation of a Cantor group. We show that for given Cantor group and minimal translation, there is a dense set of continuous sampling functions such that the spectrum of the associated operators has zero Hausdorff dimension and all spectral measures are purely singular continuous. The associated Lyapunov exponent is a continuous strictly positive function of the energy. It is possible to include a coupling constant in the model and these results then hold for every non-zero value of the coupling constant.

1. Introduction

This paper is a part of a sequence of papers devoted to the study of spectral properties of discrete one-dimensional limit-periodic Schrödinger operators. The first paper in this sequence, [7], contains results in the regime of zero Lyapunov exponents, while the present paper investigates the regime of positive Lyapunnov exponents. Our general aim is to exhibit as rich a spectral picture as possible within this class of operators. In particular, we want to show that all basic spectral types are possible and, in addition, in the case of singular continuous spectrum and pure point spectrum, we are interested in examples with positive Lyapunov exponents and examples with zero Lyapunov exponents. From this point of view, the present paper will, to the best of our knowledge for the first time, exhibit limit-periodic Schrödinger operators with purely singular continuous spectrum and positive Lyapunov exponents (whereas [7] had the first examples of limit-periodic Schrödinger operators with purely singular continuous spectrum and zero Lyapunov exponents). Examples with purely absolutely continuous spectrum have been known for a long time, dating back to works of Avron and Simon [2], Chulaevsky [4], and Pastur and Tkachenko [15, 16] in the 1980's. These examples (must) have zero Lyapunov exponents. Examples with pure point spectrum (and positive Lyapunov exponents at least at many energies in the spectrum) can be found in Pöschel's paper [17]; compare also the work of Chulaevsky and Molchanov [13] (who have examples with zero Lyapunov exponents). In the third paper of this sequence we use Pöschel's general theorem from [17] to construct limit-periodic examples with uniform pure point spectrum within our framework (actually these examples have uniform localization of eigenfunctions); see [8].

Our study is motivated by the recent paper [1], in which Avila disproves a conjecture raised by Simon; see [19, Conjecture 8.7]. That is, he has shown that it is possible to have ergodic potentials with uniformly positive Lyapunov exponents and zero-measure spectrum. The examples constructed by Avila are limit-periodic. In fact, the paper [1] proposes a novel way of looking at limit-periodic potentials. In hindsight, this way is

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quite natural and provides one with powerful technical tools. Consequently, we feel that a general study of limit-periodic Schrödinger operators may be based on this new approach and we have implemented this in [7, 8] and the present paper. We anticipate that further results may be obtained along these lines.

It has been understood since the early papers on limit-periodic Schrödinger operators, and more generally almost periodic Schrödinger operators, that these operators belong naturally to the class of ergodic Schrödinger operators, where the potentials are obtained dynamically, that is, by iterating an ergodic map and sampling along the iterates with a real-valued function; see [3, 5, 14] for general background. Indeed, taking the closure in ℓ^∞ of the set of translates of an almost periodic function on $\mathbb{Z}$ (i.e., the *hull* of the function), one obtains a compact Abelian group with a unique translation invariant probability measure (Haar measure). In particular, the shift on the hull is ergodic with respect to Haar measure and each element of the hull may be obtained by continuous sampling (using the evaluation at the origin, for example).

As pointed out by Avila, it is quite natural to take this one step further. That is, once a compact Abelian group and a minimal translation have been fixed, one is certainly not bound to sample along the orbits merely with functions that evaluate a sequence at one point. Rather, every continuous real-valued function on the group is a reasonable sampling function. While this is quite standard in the quasi-periodic case, we are not aware of any systematic use of it in the context of limit-periodic potentials before Avila's work [1].

The ability to fix the base dynamics and independently vary the sampling functions is very useful in constructing examples of potentials and operators that exhibit a certain desired spectral feature. This has been nicely demonstrated in [1] and is also the guiding principle in our present work. As mentioned above, our main motivation is to find examples of limit-periodic Schrödinger operators with prescribed spectral type. From this point of view, the singular continuity result we prove here is the main result of the paper. However, there was additional motivation to improve the zero measure result of Avila to a zero Hasudorff dimension result. Recent work of Damanik and Gorodetski [9, 10] focused on the weakly coupled Fibonacci Hamiltonian. This is an ergodic model that is not (uniformly) almost periodic. Among the results obtained in [9, 10], there is a theorem that states that the Hausdorff dimension of the spectrum, as a function of the coupling constant, is continuous at zero. That is, as the coupling constant approaches zero, the Hausdorff dimension of the spectrum approaches $\dim_H([-2, 2]) = 1$. When presenting this result, the authors of [9, 10] were asked whether this is a universal feature, which holds for all potentials. Thus, our purpose here is to show that there are indeed limit-periodic potentials such that continuity at zero coupling fails in the worst way possible, that is, the Hausdorff dimension of the spectrum is zero for all non-zero values of the coupling constant.[1]

Let us now describe the models and results in detail. We consider discrete one-dimensional ergodic Schrödinger operators acting in $\ell^2(\mathbb{Z})$ given by

$$[H^\omega_{f,T}\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n) \tag{1}$$

with

$$V_\omega(n) = f(T^n(\omega)),$$

[1] Our work was carried out right after the preprint leading to the publication [1] had been released. That version proved zero-measure and did not discuss the Hausdorff dimension issue. After we informed Avila about our results, we learned from him that he had added a remark to the final version of [1] stating that a suitable modification of his proof of zero measure yields zero Hausdorff dimension; see [1, Remark 1.1].

where ω belongs to a compact space Ω, $T : \Omega \to \Omega$ is a homeomorphism preserving an ergodic Borel probability measure μ and $f : \Omega \to \mathbb{R}$ is a continuous sampling function. It is often beneficial to study the operators $\{H^{\omega}_{f,T}\}_{\omega\in\Omega}$ as a family, as opposed to a collection of individual operators, since the spectrum and the spectral type of $H^{\omega}_{f,T}$ are always μ-almost surely independent of ω due to ergodicity. Moreover, if T is in addition minimal (i.e., all T-orbits are dense), then both the spectrum and the absolutely continuous spectrum of $H^{\omega}_{f,T}$ are independent of ω.

The Lyapunov exponent is defined as

$$L(E, T, f) = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} \log \|A_n^{(E,T,f)}(\omega)\| \, d\mu(\omega), \tag{2}$$

where $E \in \mathbb{R}$ is called the energy and $A_n^{(E,T,f)}$ is the n-step transfer matrix of (1) defined as

$$A_n^{(E,T,f)}(\omega) = S_{n-1} \dots S_0, \text{ where } S_i = \begin{pmatrix} E - f(T^i(\omega)) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

By the Ishii-Pastur-Kotani theorem, the almost sure absolutely continuous spectrum of $H^{\omega}_{f,T}$ is given by the essential closure of the set of energies where the Lyapunov exponent vanishes.

Next we make the spaces and homeomorphisms of especial interest to us explicit.

Definition 1.1. *Ω is called a Cantor group if it is an infinite totally disconnected compact Abelian topological group.*

Definition 1.2. *Let Ω be a Cantor group. For $\omega_1 \in \Omega$, let $T : \Omega \to \Omega$ be the translation by ω_1, that is, $T(\omega) = \omega_1 \cdot \omega$. T is called minimal if $\{T^n(\omega) : n \in \mathbb{Z}\}$ is dense in Ω for every $\omega \in \Omega$.*

We will restrict our attention to the case where Ω is a Cantor group and T is a minimal translation. As mentioned above, the operators $H^{\omega}_{f,T}$ have a common spectrum which we will denote by $\Sigma(f)$.

Here is our main result:

Theorem 1.3. *Suppose Ω is a Cantor group and T is a minimal translation on Ω. Then these exists a dense set $\mathcal{F} \subset C(\Omega, \mathbb{R})$ such that for every $f \in \mathcal{F}$ and every $\lambda \neq 0$, the following statements hold true: $\Sigma(\lambda f)$ has zero Hausdorff dimension, $H^{\omega}_{\lambda f,T}$ has purely singular continuous spectrum for every $\omega \in \Omega$, and $E \mapsto L(E, T, \lambda f)$ is a positive continuous function.*

The proof of this theorem is based on the constructions in [1]. We make several modifications to these constructions to better control the size of the spectrum and to ensure that the potentials we construct are Gordon potentials. The latter property then implies the absence of point spectrum, which in turn yields singular continuity since the absence of absolutely continuous spectrum already follows from zero measure spectrum.

Let us state the Gordon property as a separate result.

Definition 1.4. *A bounded map $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there exist positive integers $q_i \to \infty$ such that*

$$\max_{1\le n\le q_i} |V(n) - V(n \pm q_i)| \le i^{-q_i}$$

for every $i \geq 1$.

Clearly, if V is a Gordon potential, so is λV for every $\lambda \in \mathbb{R}$. A key part in proving Theorem 1.3 is to establish the following result:

Theorem 1.5. *Suppose Ω is a Cantor group. Then these exists a dense set $\mathcal{F} \subset C(\Omega, \mathbb{R})$ such that for every $f \in \mathcal{F}$, every minimal translation $T : \Omega \to \Omega$, every $\omega \in \Omega$, and every $\lambda \neq 0$, $\lambda f(T^n(\omega))$ is a Gordon potential.*

2. Preliminaries

2.1. **Hausdorff Measures and Dimensions.** For our relatively restricted purposes, we will simply recall the definition of Hausdorff measures and Hausdorff dimension in this subsection. We refer the reader to [18] for more information.

Definition 2.1. *Let $A \subseteq \mathbb{R}$ be a subset. A countable collection of intervals $\{b_n\}_{n=1}^{\infty}$ is called a δ-cover of A if $A \subset \bigcup_{n=1}^{\infty} b_n$ with $|b_n| < \delta$ for all n's.* (*Here, $|\cdot|$ denotes Lebesgue measure, and we will adopt this notation throughout the paper.*)

Definition 2.2. *Let $\alpha \in \mathbb{R}$. For any subset $A \subseteq \mathbb{R}$, the α-dimensional Hausdorff measure of A is defined as*

$$h^{\alpha}(A) = \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum_{n=1}^{\infty} |b_n|^{\alpha}. \tag{4}$$

The quantity $h^{\alpha}(A)$ is well defined as an element of $[0, \infty]$ since $\inf_{\delta\text{-covers}} \sum_{n=1}^{\infty} |b_n|^{\alpha}$ is monotonically increasing as δ decreases to zero and therefore the limit in (4) exists. Restricted to the Borel sets, h^1 coincides with Lebesgue measure and h^0 is the counting measure. If $\alpha < 0$, we always have $h^{\alpha}(A) = \infty$ for any $A \neq \emptyset$, while if $\alpha > 1$, $h^{\alpha}(\mathbb{R}) = 0$.

It is not hard to see that for every $A \subseteq \mathbb{R}$, there is a unique $\alpha \in [0, 1]$, called the Hausdorff dimension $\dim_H(A)$ of A, such that $h^{\beta}(A) = \infty$ for every $\beta < \alpha$ and $h^{\beta}(A) = 0$ for every $\beta > \alpha$. In particular, every $A \subseteq \mathbb{R}$ with $|A| > 0$ must have $\dim_H(A) = 1$.

2.2. **Minimal Translations of Cantor Groups and Limit-Periodic Potentials.** In this subsection we recall how the one-to-one correspondence between hulls of limit-periodic sequences and potential families generated by minimal translations of Cantor groups and continuous sampling functions exhibited by Avila in [1] arises.

Definition 2.3. *Let $S : \ell^{\infty}(\mathbb{Z}) \to \ell^{\infty}(\mathbb{Z})$ be the shift operator, $(SV)(n) = V(n + 1)$. A two-sided sequence $V \in \ell^{\infty}(\mathbb{Z})$ is called periodic if its S-orbit is finite and it is called limit-periodic if it belongs to the closure of the set of periodic sequences. If V is limit-periodic, the closure of its S-orbit is called the hull and denoted by* hull_V.

The first lemma (see [1, Lemma 2.1]) shows how one can write the elements of the hull of a limit-periodic function in the form

$$V_{\omega}(n) = f(T^n(\omega)), \quad \omega \in \Omega,\ n \in \mathbb{Z} \tag{5}$$

with a minimal translation T of a Cantor group and a sampling function $f \in C(\Omega, \mathbb{R})$:

Lemma 2.4. *Suppose V is limit-periodic. Then, $\Omega := \mathrm{hull}_V$ is compact and has a unique topological group structure with identity V such that $\mathbb{Z} \ni k \mapsto S^k V \in \mathrm{hull}_V$ is a homomorphism. Moreover, the group structure is Abelian and there exist arbitrarily small compact open neighborhoods of V in hull_V that are finite index subgroups.*

In particular, $\Omega = \mathrm{hull}_V$ is a Cantor group, $T = S|_{\Omega}$ is a minimal translation, and every element of Ω may be written in the form (5) with the continuous function $f(\omega) = \omega(0)$.

The second lemma (see [1, Lemma 2.2]) addresses the converse:

Lemma 2.5. *Suppose Ω is a Cantor group, $T : \Omega \to \Omega$ is a minimal translation, and $f \in C(\Omega, \mathbb{R})$. Then, for every $\omega \in \Omega$, the element V_ω of $\ell^\infty(\mathbb{Z})$ defined by* (5) *is limit-periodic and we have* $\mathrm{hull}_{V_\omega} = \{V_{\tilde{\omega}}\}_{\tilde{\omega} \in \Omega}$.

These two lemmas show that a study of limit-periodic potentials can be carried out by considering potentials of the form (5) with a minimal translation T of a Cantor group Ω and a continuous sampling function f. As shown for the first time in the context of limit-periodic potentials by Avila in [1], it is often advantageous to fix Ω and T and to vary f.

2.3. **Periodic Sampling Functions, Potentials, and Schrödinger Operators.** In this subsection we discuss the periodic case. For example, which sampling functions $f \in C(\Omega, \mathbb{R})$ give rise to periodic potentials for some or all (ω, T)? Moreover, what can then be said about the associated Schrödinger operators?

Definition 2.6. *Suppose Ω is a Cantor group and $T : \Omega \to \Omega$ is a minimal translation. We say that a sampling function $f \in C(\Omega, \mathbb{R})$ is n-periodic with respect to T if $f(T^n(\omega)) = f(\omega)$ for every $\omega \in \Omega$.*

Proposition 2.7. *Let $f \in C(\Omega, \mathbb{R})$. If $f(T^{n_0+m}(\omega_0)) = f(T^m(\omega_0))$ for some $\omega_0 \in \Omega$, some minimal translation $T : \Omega \to \Omega$ and every $m \in \mathbb{Z}$, then for every minimal translation $\tilde{T} : \Omega \to \Omega$, f is n_0-periodic with respect to $\tilde{T}$.*

Proof. Let $\varphi : \Omega \to \ell^\infty(\mathbb{Z})$, $\varphi(\omega) = (f(T^n(\omega)))_{n \in \mathbb{Z}}$. Since T is minimal, the closure of $\{T^n(\omega_0) : n \in \mathbb{Z}\}$ is Ω. By Lemma 2.5 we have $\varphi(\Omega) = \mathrm{hull}(\varphi(\omega_0))$. Since $f(T^{n_0+m}(\omega_0)) = f(T^m(\omega_0))$ for any $m \in \mathbb{Z}$, $\mathrm{hull}(\varphi(\omega_0))$ is a finite set. Then for any $\omega \in \Omega$, $(f(T^n(\omega)))_{n \in \mathbb{Z}}$ is some element in $\mathrm{hull}(\varphi(\omega_0))$. Since every element in $\mathrm{hull}(\varphi(\omega_0))$ is n_0-periodic, $(f(T^n(\omega)))_{n \in \mathbb{Z}}$ is n_0-periodic. This shows that f is n_0-periodic with respect to T. That is, we have $f(T^{n_0+m}(\omega)) = f(T^m(\omega))$ for every $\omega \in \Omega$ and $m \in \mathbb{Z}$.

Assume T is the minimal translation by ω_1 and let $\tilde{T}$ be another minimal translation by ω_2. By the previous analysis, we have $f(\omega_1^{n_0+m} \cdot \omega) = f(\omega_1^m \cdot \omega)$ for every $m \in \mathbb{Z}$ and every $\omega \in \Omega$. If ω_2 is equal to ω_1^q for some integer q, obviously we have $f(\tilde{T}^{n_0}(\omega)) = f((\omega_1^q)^{n_0} \cdot \omega) = f(\omega)$ for any $\omega \in \Omega$. If not, since $\{\omega_1^n : n \in \mathbb{Z}\}$ is dense in Ω (this follows from the minimality of T), we have $\lim_{k \to \infty} \omega_1^{n_k} = \omega_2$, and then $f(\omega_2^{n_0} \cdot \omega) = \lim_{k \to \infty} f((\omega_1^{n_k})^{n_0} \cdot \omega) = f(\omega)$. The result follows. $\square$

The above proposition tells us that the periodicity of f is independent of T. Thus we may say f is n-periodic without making a minimal translation explicit.

Next we recall from [1] how periodic sampling functions in $C(\Omega, \mathbb{R})$ can be constructed. Given a Cantor group Ω, a compact subgroup Ω_0 with finite index (such subgroups can be found in any neighborhood of the identity element; see above), and $f \in C(\Omega, \mathbb{R})$, we can define a periodic $f_{\Omega_0} \in C(\Omega, \mathbb{R})$ by

$$f_{\Omega_0}(\omega) = \int_{\Omega_0} f(\omega \cdot \tilde{\omega}) \, d\mu_{\Omega_0}(\tilde{\omega}).$$

Here, μ_{Ω_0} denotes Haar measure on Ω_0. This shows that the set of periodic sampling functions is dense in $C(\Omega, \mathbb{R})$. Moreover, as already noted in [1], there exists a decreasing sequence of Cantor subgroups Ω_k with finite index n_k such that $\bigcap \Omega_k = \{e\}$, where e is the identity element of Ω. Let P_k be the set of sampling functions defined on Ω/Ω_k, that is, the elements in P_k are n_k-periodic potentials. Denote by P the set of all periodic sampling functions. Then, we have $P_k \subset P_{k+1}$ (which implies $n_k \mid n_{k+1}$) and $P = \bigcup P_k$.

Proposition 2.8. *Let f be p-periodic. Then, for every $\omega \in \Omega$,*

$$
\begin{aligned}
L(E, T, f) &= \lim_{m\to\infty} \frac{1}{m} \log \|A_m^{(E,T,f)}(\omega)\| \\
&= \frac{1}{p} \log \rho(A_p^{(E,T,f)}(e)),
\end{aligned} \tag{6}
$$

where $\rho(A_p^{(E,T,f)}(e))$ is the spectral radius of $A_p^{(E,T,f)}(e)$. In particular, if restricted to periodic sampling functions, the Lyapunov exponent is a continuous function of both the energy E and the sampling function.

Proof. If f is p-periodic, as in the proof of Proposition 2.7, for every ω, $(f(T^n(\omega)))_{n\in\mathbb{Z}}$ is some element of the orbit of $(f(T^n(e)))_{n\in\mathbb{Z}}$, and so its monodromy matrix (i.e., the transfer matrix over one period) is a cyclic permutation of the monodromy matrix associated with $f(T^n(e))$. Thus $\mathrm{Tr}A_p^{(E,T,f)}(\omega)$ is independent of ω, and since $\det A_p^{(E,T,f)}(\omega) = 1$, we can conclude that the eigenvalues of $A_p^{(E,T,f)}(\omega)$ are independent of ω. So the logarithm of the spectral radius of $A_p^{(E,T,f)}(\omega)$ is independent of ω and (6) follows. The continuity statement follows readily. □

Lemma 2.9. *Let $f_n \in C(\Omega, \mathbb{R})$ be a sequence of periodic sampling functions converging uniformly to $f_\infty \in C(\Omega, \mathbb{R})$. Assume $\lim_{n\to\infty} L(E, T, f_n)$ exists for every E and the convergence is uniform. Then we have that $L(E, T, f_\infty)$ coincides with $\lim_{n\to\infty} L(E, T, f_n)$ everywhere.*

Proof. Since $\lim_{n\to\infty} L(E, T, f_n)$ exists everywhere, from [1, Lemma 2.5], we have $L(E, T, f_n) \to L(E, T, f_\infty)$ in L^1_{loc}. So $L(E, T, f_\infty)$ coincides with $\lim_{n\to\infty} L(E, T, f_n)$ almost everywhere. From Proposition 2.8, $L(E, T, f_n)$ is a continuous function, and by uniform convergence, we have that $\lim_{n\to\infty} L(E, T, f_n)$ is also a continuous function. Since $L(E, T, f_\infty)$ is a subharmonic function (cf. [6, Theorem 2.1]), we get that $L(E, T, f_\infty) = \lim_{n\to\infty} L(E, T, f_n)$ for every E. The statement follows. □

To conclude this subsection on the periodic case, we state two lemmas. The first is well known and the second is [1, Lemma 2.4].

Lemma 2.10. *Let $f \in C(\Omega, \mathbb{R})$ be p-periodic.*
(i). The spectrum of $H^\omega_{f,t}$ is purely absolutely continuous for every $\omega \in \Omega$ and $\Sigma(f)$ is made of p bands (compact intervals whose interiors are disjoint).
(ii). $\Sigma(f) = \{E \in \mathbb{R} : L(E, T, f) = 0\}$.

Lemma 2.11. *Let $f \in C(\Omega, \mathbb{R})$ be p-periodic.*
(i). The Lebesgue measure of each band of $\Sigma(f)$ is at most $\frac{2\pi}{p}$.
(ii). Let $C \geq 1$ be such that for every $E \in \Sigma(f)$, there exist $\omega \in \Omega$ and $k \geq 1$ such that $\|A_k^{(E,T,f)}(\omega)\| \geq C$. Then, $|\Sigma(f)| \leq \frac{4\pi p}{C}$.

3. Proof of the Theorems

Assume Ω and T are given. For convenience, we write $A_n^{(E,f)}(\omega) = A_n^{(E,f,T)}(\omega)$, $A_n^{(E,f)} = A_n^{(E,f,T)}(e)$, and $L(E, f) = L(E, T, f)$. Since $T : \Omega \to \Omega$ is a minimal translation, the homomorphism $\mathbb{Z} \to \Omega$, $n \to T^n e$ is injective with dense image in Ω, and we can write $f(n) = f(T^n(e))$ without any conflicts.

We need two more lemmas before proving our theorems. More precisely, we will make further use of the constructions which play central roles in the proof of these two lemmas.

Lemma 3.1. *Let B be an open ball in $C(\Omega, \mathbb{R})$, let $F \subset P \cap B$ be finite, and let $0 < \varepsilon < 1$. Then there exists a sequence $F_K \subset P \cap B$ such that*
(i). $L(E, \lambda F_K) > 0$ whenever $\varepsilon \leq |\lambda| \leq \varepsilon^{-1}$, $E \in \mathbb{R}$,
(ii). $L(E, \lambda F_K) \to L(E, \lambda F)$ uniformly on compacts (as functions of $(E, \lambda) \in \mathbb{R}^2$).

This is [1, Lemma 3.1]. As in [1], we use the notation

$$L(E, \lambda F) = \frac{1}{\#F} \sum_{f \in F} L(E, T, \lambda f),$$

where F is a finite family of sampling functions (with multiplicities!) and $\lambda \in \mathbb{R}$. The proof of this lemma is constructive. We will describe this construction explicitly in the proof of Theorem 1.3 for the reader's convenience.

Lemma 3.2. *Suppose B is an open ball in $C(\Omega, \mathbb{R})$ and $F \subset P \cap B$ is a finite family of sampling functions. Then for every $N \geq 2$ and K sufficiently large, there exists $F_K \subset P_K \cap B$ such that*
(i). $L(E, \lambda F_K) \to L(E, \lambda F)$ uniformly on compacts (as functions of $(E, \lambda) \in \mathbb{R}^2$).
(ii). The diameter of F_K is at most $n_K^{-N/2}$.

This lemma is a variation of [1, Lemma 3.2]. We will prove this lemma using suitable modifications of Avila's arguments. Some of these modifications, which will later enable us to prove the Gordon property, are not apparent from the statement of the lemma. We will give detailed arguments for the modified parts of the proof and refer the reader to [1] for the parts that are analogous.

Proof of Lemma 3.2. Assume that $F = \{f_1, f_2, \ldots, f_m\} \subset C(\Omega, \mathbb{R})$ is a finite family of n_k-periodic sampling functions with $n_k \geq 2$, and let $K > k$ be large enough. We construct $F_K^{\vec{t}}$ as follows. Let $n_K = mn_k r + d$, $0 \leq d \leq mn_k - 1$. Let $I_j = [jn_k, (j+1)n_k - 1] \subset \mathbb{Z}$ and let $0 = j_0 < j_1 < \cdots < j_{m-1} < j_m = n_K/n_k$ be a sequence such that $j_{i+1} - j_i = r + 1$ when $0 \leq i < d/n_k$ and $j_{i+1} - j_i = r$ when $d/n_k \leq i \leq m - 1$. Define an n_K-periodic f as follows. For $0 \leq l \leq n_K - 1$, let j be such that $l \in I_j$ and let i be such that $j_{i-1} \leq j < j_i$ and then let $f(l) = f_i(l)$. Next, for any sequence $\vec{t} = (t_1, t_2, \ldots, t_m)$ with $t_i \in \{0, 1, \ldots, r-1\}$, we define an n_K-periodic $f_K^{\vec{t}}$ as follows. If $j = j_i - 1$ for some $1 \leq i < m$, we let $f_K^{\vec{t}}(l) = f(l) + r^{-N} t_i$, and if $j = j_m - 2$, we let $f_K^{\vec{t}}(l) = f(l) + r^{-N} t_m$. Otherwise we let $f_K^{\vec{t}}(l) = f(l)$. Let $F_K^{\vec{t}}$ be the family consisting of all $f_K^{\vec{t}}$'s. The statement (ii) is clear for large K. (Note: in [1], Avila's construction is such that if $j = j_i - 1$ for some $1 \leq i \leq m$, then $f_K^{\vec{t}}(l) = f(l) + r^{-20} t_i$; otherwise, $f_K^{\vec{t}}(l) = f(l)$.)

For fixed E and λ, we let $A_{n_K}^{(E, \lambda f_K^{\vec{t}})} = C^{(t_m, m)} B^{(m)} \ldots C^{(t_1, 1)} B^{(1)}$, where $B^{(i)} = (A_{n_k}^{(E, \lambda f_i)})^{j_i - j_{i-1} - 1}$, $1 \leq i \leq m-1$ and $B^{(m)} = (A_{n_k}^{(E, \lambda f_m)})^{j_m - j_{m-1} - 2}$, and $C^{(t_i, i)} = A_{n_k}^{(E - \lambda r^{-N} t_i, \lambda f_i)}$, $1 \leq i \leq m-1$ and $C^{(t_m, m)} = A_{n_k}^{(E, \lambda f_m)} A_{n_k}^{(E - \lambda r^{-N} t_m, \lambda f_m)}$. When E and λ are in a compact set, the norm of the $C^{(t_i, i)}$-type matrices is bounded as r grows, while the norm of the $B^{(i)}$-type matrices may get large.

Notice that our perturbation here is $r^{-N} t$ (as opposed to Avila's $r^{-20} t$ perturbation in [1, Lemma 3.2]), so [1, Claim 3.7] should be replaced by the following version:

"Let s_j be the most contracted direction of $\hat{B}^{(j)}$ and let u_j be the image under $\hat{B}^{(j)}$ of the most expanded direction. Call $\vec{t}$ j-nice, $1 \leq j \leq d$, if the angle between $\hat{C}^{(j)} u_j$ and S_{j+1} (less than π) is at least r^{-3N} with the convention that $j+1 = 1$ for $j = d$. Let r be sufficiently large, and let $\vec{t}$ be j-nice. If z is a non-zero vector making an angle at least r^{-4N} with s_j,

then $z' = \hat{C}^{(j)}\hat{B}^{(j)}z$ makes an angle at least r^{-4N} with S_{j+1} and $\|z'\| \geq \|\hat{B}^{(j)}\|r^{-5N}\|z\|$."

The proof of [1, Claim 3.7] can be applied to get the above version of the claim with the corresponding quantitative modification. Moreover, we have also made a little shift in the perturbation, so $C^{(t_m,m)} = A_{n_k}^{(E,\lambda f_m)} A_{n_k}^{(E-\lambda r^{-N}t_m,\lambda f_m)}$, while Avila's $C^{(t_m,m)} = A_{n_k}^{(E-\lambda r^{-20}t_m,\lambda f_m)}$. [1, Claim 3.8] still holds, but Avila's proof of [1, Claim 3.8] cannot be applied directly. To this end we prove the following claim:

Claim 3.3. *For any $M \in \mathrm{SL}(2,R)$, there are $m_1, m_2 \in (0,\infty)$ with the following property. Suppose A and B are two vectors in $\mathbb{R}^2$, and $\Delta\theta$ is the angle between A and B with $0 < \Delta\theta \leq \pi$. Let $\Delta\tilde{\theta}$ be the angle between MA and MB (again so that $0 < \Delta\tilde{\theta} \leq \pi$). Then, $m_1\Delta\theta \leq \Delta\tilde{\theta} \leq m_2\Delta\theta$.*

Proof. By the singular value decomposition (see [11, Theorem 2.5.1]), there exist O_1 and O_2 in $\mathrm{SO}(2,R)$ such that $M = O_1 S O_2$, where S is a diagonal matrix. Since O_1 and O_2 are rotations on $\mathbb{R}^2$, it is sufficient to consider

$$S = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix}.$$

Without loss of generality, assume $\mu_1 \geq 1$. Let $A = (a,b)^t$ (t denotes the transpose of vectors) and $B = (c,d)^t$ be two normalized vectors, and let θ_A and θ_B be the argument of A and the argument of B respectively. Let $\tilde{A} = SA = (a\mu_1, b/\mu_1)^t$ with the argument $\theta_{\tilde{A}}$ and $\tilde{B} = SB = (c\mu_1, d/\mu_1)^t$ with the argument $\theta_{\tilde{B}}$.

We adopt the following notation for convenience. Let I, II,III,IV denote one of two vectors in the first quadrant (including $\{(x,0) : x \geq 0\}$), the second quadrant (including $\{(0,y) : y > 0\}$), the third quadrant (including $\{(x,0) : x < 0\}$) and the fourth quadrant (including $\{(0,y) : y < 0\}$), respectively. Then (I,I) denotes that both two vectors are in the first quadrant, (I,II) denotes that one vector is in the first quadrant while the other is in the second quadrant, and so on.

We will need the following observation:

$$0 < \theta_1, \theta_2 < \pi/2 \text{ and } \tan\theta_1 \geq \frac{\tan\theta_2}{\mu_1^2} \quad \Rightarrow \quad \theta_1 \geq \frac{\theta_2}{4\mu_1^2}. \tag{7}$$

Indeed, since $\tan\theta_1 \geq \frac{1}{\mu_1^2}\tan\theta_2 \geq \frac{1}{2\mu_1^2}\theta_2$ and $0 < \frac{\theta_2}{2\mu_1^2} < 1$, we have

$$\theta_1 \geq \arctan\frac{\theta_2}{2\mu_1^2} = \frac{\theta_2}{2\mu_1^2} - (\frac{\theta_2}{2\mu_1^2})^3/3 + O((\frac{\theta_2}{2\mu_1^2})^5) \geq \frac{\theta_2}{4\mu_1^2}.$$

For the proof of Claim 3.3, we consider two cases.

Case 1. $\pi/2 \leq \Delta\theta \leq \pi$. Here A and B cannot be in the same quadrant. Notice that the impact of S on vectors is to move them closer to the x-axis and keep them in the same quadrant. Thus, for the subcases (I,II), (I,III), (II,IV) and (III,IV), we can easily conclude that $\Delta\theta/2 \leq \Delta\tilde{\theta} \leq 2\Delta\theta$. There are two subcases left, (I,IV) and (II,III). We will discuss (I,IV); the method can be readily adapted to (II,III). For (I,IV), if $\theta_A = 0$ and $\theta_B = 3\pi/2$, then $\theta_{\tilde{A}}$ and $\theta_{\tilde{B}}$ are also 0 and $3\pi/2$ respectively, and so $\Delta\tilde{\theta} = \Delta\theta$; if not, without loss of generality, assume that A is in the first quadrant with $\pi/4 \leq \theta_A < \pi/2$, then $\tan\theta_{\tilde{A}} = \frac{b}{a\mu_1^2} = \frac{\tan\theta_A}{\mu_1^2}$, and by (7), we have

$$\Delta\tilde{\theta} \geq \theta_{\tilde{A}} \geq \frac{\theta_A}{4\mu_1^2} \geq \frac{\Delta\theta}{16\mu_1^2}$$

($\theta_A \geq \Delta\theta/4$ since $\theta_A \geq \pi/4$) and then $\frac{\Delta\theta}{16\mu_1^2} \leq \Delta\tilde{\theta} \leq 2\Delta\theta$.

Case 2. $0 < \Delta\theta < \pi/2$. In this case, (I, III) and (II, IV) are impossible. We will divide the following proof into three parts.
(1). We discuss (I, I) here; the argument may be readily adapted to (II, II), (III, III), and (IV, IV). Without loss of generality, assume $\Delta\theta = \theta_A - \theta_B$, then we get

$$\tan \Delta\tilde{\theta} = \frac{\mu_1^2(bc - ad)}{bd + \mu_1^4 ac} \geq \frac{\tan \Delta\theta}{\mu_1^2},$$

and by (7), we get $\frac{\Delta\theta}{4\mu_1^2} \leq \Delta\tilde{\theta}$. Similarly, we will get $\Delta\tilde{\theta} \leq 4\mu_1^2 \Delta\theta$ since $\tan \Delta\tilde{\theta} \leq \mu_1^2 \tan \Delta\theta$, and so $\frac{\Delta\theta}{4\mu_1^2} \leq \Delta\tilde{\theta} \leq 4\mu_1^2 \Delta\theta$ follows.
(2). We discuss (I, IV) here; an adaptation handles (II, III). Without loss of generality, assume $\theta_A \geq \Delta\theta/2$. Obviously, we have $\Delta\tilde{\theta} \leq \Delta\theta$. Conversely, we have $\frac{\Delta\theta}{16\mu_1^2} \leq \Delta\tilde{\theta}$ (it is essentially the same as (I, IV) in *Case 1*), and so $\frac{\Delta\theta}{16\mu_1^2} \leq \Delta\tilde{\theta} \leq \Delta\theta$ follows.
(3). We discuss (I, II) here, and the method can be applied to (III, IV). Obviously, we have $\Delta\theta \leq \Delta\tilde{\theta}$. Without loss of generality, assume that A is in the first quadrant and makes an angle h_A with the y-axis and that B is in the second quadrant and makes an angle h_B with the y-axis. Clearly, $\Delta\theta = h_A + h_B$. Let $h_{\tilde{A}}$ and $h_{\tilde{B}}$ be the angle between the y-axis and $\tilde{A}$ and the angle between the y-axis and $\tilde{B}$, respectively. By (7), we conclude that $h_{\tilde{A}} \leq 4\mu_1^2 h_A$ since $\tan h_{\tilde{A}} = \mu_1^2 \tan h_A$. Similarly, we get $h_{\tilde{B}} \leq 4\mu_1^2 h_B$. So it follows that $\Delta\theta \leq \Delta\tilde{\theta} = h_{\tilde{A}} + h_{\tilde{B}} \leq 4\mu_1^2(h_A + h_B) = 4\mu_1^2 \Delta\theta$.

Through the above analysis, we see that $\frac{\Delta\theta}{16\mu_1^2} \leq \Delta\tilde{\theta} \leq 16\mu_1^2 \Delta\theta$, concluding the proof of Claim 3.3. □

By this claim, we can modify the last paragraph of the proof of [1, Claim 3.8] as stated below and then our lemma follows.

"If r sufficiently large, we conclude that for every $0 \leq l \leq r - 2$, there exists a rotation $R_{l,j}$ of angle θ_j with $r^{-2.5N} < \theta_j < r^{-0.3N}$ such that $C^{(l+1,i_j)} u_j = R_{l,j} C^{(l,i_j)} u_j$. It immediately follows that there exists at most one choice of $0 \leq t_{i_j} \leq r - 1$ such that $C^{(t_{i_j}, i_j)} u_j$ has angle at most r^{-3N} with s_{j+1}, as desired."

We would like to explain how to obtain the statement described in the paragraph above. If r is sufficiently large, it is not hard to conclude that for every $0 \leq l \leq r - 2$, there exists a rotation $\tilde{R}_{l,j}$ of angle $\tilde{\theta}_j$ with $r^{-2N} < \tilde{\theta}_j < r^{-0.5N}$ such that $A_{n_k}^{(E - \lambda r^{-N}(l+1), \lambda f_{i_j})} u_j = \tilde{R}_{l,j} A_{n_k}^{(E - \lambda r^{-N} l, \lambda f_{i_j})} u_j$. If $i_d = m$, we have

$$C^{(l+1,m)} u_m = A_{n_k}^{(E,\lambda f_m)} A_{n_k}^{(E - \lambda r^{-N}(l+1), \lambda f_m)} u_m \tag{8}$$
$$= A_{n_k}^{(E,\lambda f_m)} \tilde{R}_{l,m} A_{n_k}^{(E - \lambda r^{-N}(l), \lambda f_m)} u_m.$$

Since $A_{n_k}^{(E,\lambda f_m)} \in \mathrm{SL}(2, R)$ is independent of r, we can apply Claim 3.3 to (8) so that we have

$$\begin{aligned} C^{(l+1,m)} u_m &= A_{n_k}^{(E,\lambda f_m)} \tilde{R}_{l,m} A_{n_k}^{(E - \lambda r^{-N}(l), \lambda f_m)} u_m \\ &= R_{l,m} A_{n_k}^{(E,\lambda f_m)} A_{n_k}^{(E - \lambda r^{-N}(l), \lambda f_m)} u_m \\ &= R_{l,m} C^{(l,m)} u_m, \end{aligned}$$

where $R_{l,m}$ is a rotation of angle θ_m with $r^{-2.5N} < \theta_m < r^{-0.3N}$. Then the above paragraph follows. □

Recall the definition of a Gordon potential given in Definition 1.4. The importance of Gordon potentials lies in the following lemma, which (in a slightly weaker form) first appeared in [12].

Lemma 3.4 (Gordon Lemma)**.** *Suppose V is a Gordon potential. Then the Schrödinger operator with potential V has no eigenvalues.*

Now we can give the

Proof of Theorem 1.3. Given a p_0-periodic $f \in C(\Omega, \mathbb{R})$ and $0 < \varepsilon_0 < 1$, consider $B_{\varepsilon_0}(f) \subset C(\Omega, \mathbb{R})$. (We will work within this ball. The denseness of periodic potentials then implies the denseness of our constructed limit-periodic potentials.) Let N from Lemma 3.2 be 2. Let $\varepsilon_1 = \frac{\varepsilon_0}{10}$. By Lemma 3.1, there exists a finite family $F_1 = \{f_1, f_2, \dots f_{m_1}\}$ of p_1-periodic sampling functions such that $F_1 \subset B_{\varepsilon_0}(f)$ and $L(E, \lambda F_1) > \delta_1$ for some $0 < \delta_1 < 1$ whenever $\varepsilon_1 < |\lambda| < \frac{1}{\varepsilon_1}$ and $E \in \mathbb{R}$ (note that $L(E, \lambda f_i) \geq 1$ if $|E| \geq \|\lambda f_i\| + 4$). Our constructions start with F_1 and we will divide them into several steps.

Construction 1. First, we will apply Lemma 3.1 to F_1 in order to enlarge the range of λ's. Let $\varepsilon_2 = \frac{min\{\varepsilon_1, \delta_1\}}{10}$. Then, there exists a finite family of $\tilde{p}_1$-periodic potentials $\tilde{F}_1 = \{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{\tilde{m}_1}\} \subset B_{\varepsilon_0}(f)$ such that

$$L(E, \lambda \tilde{F}_1) > \tilde{\delta}_1$$

for some $0 < \tilde{\delta}_1 < 1$ whenever $\forall \varepsilon_2 < |\lambda| < \frac{1}{\varepsilon_2}$ and $E \in \mathbb{R}$, and

$$\left| L(E, \lambda \tilde{F}_1) - L(E, \lambda F_1) \right| < \frac{\varepsilon_2}{2} \tag{9}$$

whenever $|E| < \frac{1}{\varepsilon_2}$ and $|\lambda| < \frac{1}{\varepsilon_2}$.

Explicitly, the construction of $\tilde{F}_1$ follows from the proof of [1, Claim 3.1]. For very large $\tilde{p}_1 > p_1$ (it must obey $p_1 | \tilde{p}_1$), choose $N_1(\tilde{p}_1)$ such that if $|E| < \frac{1}{\varepsilon_2}$, $|\lambda| \leq \frac{1}{\varepsilon_2}$, $f_i \in F_1$ and a $\tilde{p}_1$-periodic potential $\tilde{f}$ which is $\frac{2p_1+1}{N_1(\tilde{p}_1)}$ close to f_i then $|L(E, \lambda \tilde{f}) - L(E, \lambda f_i)| < \frac{\varepsilon_2}{2}$, since the Lyapunov exponent is continuous for periodic potentials (see Proposition 2.8).

For $1 \leq j \leq 2p_1 + 1$, we define $\tilde{p}_1$-periodic potentials $\tilde{f}^{(i,j)}$ by $\tilde{f}^{(i,j)}(n) = f_i(n), 0 \leq n \leq \tilde{p}_1 - 2$ and $\tilde{f}^{(i,j)}(\tilde{p}_1 - 1) = f_i(\tilde{p}_1 - 1) + \frac{j}{N_1(\tilde{p}_1)}$. By [1, Claim 3.4], there exists j_0 such that the spectrum of $\tilde{f}^{(i,j_0)}$ has exactly $\tilde{p}_1$ components, that is, all gaps of its spectrum are open. For convenience, we write $\tilde{f}^{(i)} = \tilde{f}^{(i,j_0)}$. So there exists $h = h(F_1, \tilde{p}_1, \varepsilon_2) > 0$ such that for any $f_i \in F_1$ and $\varepsilon_2 \leq |\lambda| \leq \frac{1}{\varepsilon_2}$, $\Sigma(\lambda \tilde{f}^{(i)})$ has $\tilde{p}_1$ components and the Lebesgue measure of the smallest gap is at least h. Choose an integer $N_2(\tilde{p}_1)$ with $N_2(\tilde{p}_1) > \frac{4\pi}{\varepsilon_2 h \tilde{p}_1}$.

For $0 \leq l \leq N_2(\tilde{p}_1)$, let $\tilde{f}^{(i,l)} = \tilde{f}^{(i)} + \frac{4\pi l}{\varepsilon_2 \tilde{p}_1 N_2(\tilde{p}_1)}$. Then $\tilde{F}_1$ is just the family obtained by collecting the $\tilde{f}^{(i,l)}$ for different $f_i \in F_1$ and $0 \leq l \leq N_2(\tilde{p}_1)$. Order $\tilde{F}_1$ as $\tilde{F}_1 = \{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{\tilde{m}_1}\}$ such that $\tilde{f}_1 = \tilde{f}^{(1,0)}$ and $\tilde{f}_{\tilde{m}_1} = \tilde{f}^{(1,1)}$. We can also assume that $N_2(\tilde{p}_1)$ was chosen large enough, so that we have $\|\tilde{f}_{\tilde{m}_1} - \tilde{f}_1\| = \frac{4\pi}{\varepsilon_2 \tilde{p}_1 N_2(\tilde{p}_1)} < 1/3$ (this will be used to conclude that our limit-periodic potentials are Gordon potentials).

Construction 2. Applying Lemma 3.2 to $\tilde{F}_1$, there exists a finite family of p_2-periodic potentials $F_2 = \{f_2^{\vec{t}_1}, f_2^{\vec{t}_2}, \dots, f_2^{\vec{t}_{m_2}}\}$ such that

$$F_2 \subset B_{p_2^{-2}} \subset B_{\varepsilon_2} \subset B_{\varepsilon_0}(f)$$

and

$$L(E, \lambda F_2) > \delta_2$$

for some $0 < \delta_2 < 1$ whenever $\varepsilon_2 < |\lambda| < \frac{1}{\varepsilon_2}$ and $E \in \mathbb{R}$, and

$$\left| L(E, \lambda F_2) - L(E, \lambda \tilde{F}_1) \right| < \frac{\varepsilon_2}{2} \tag{10}$$

whenever $|E|, |\lambda| < \frac{1}{\varepsilon_2}$. From (9) and (10), we have

$$|L(E, \lambda F_2) - L(E, \lambda F_1)| < \varepsilon_2$$

for $|E|, |\lambda| < \frac{1}{\varepsilon_2}$.

Explicitly, we construct F_2 as follows (cf. the proof of Lemma 3.2). Let p_2 large and $p_2 = \tilde{m}_1 \tilde{p}_1 r_2 + d$, $0 \le d \le \tilde{m}_1 \tilde{p}_1 - 1$. Let $I_j = [j\tilde{p}_1, (j+1)\tilde{p}_1 - 1] \subset \mathbb{Z}$ and let $0 = j_0 < j_1 < \cdots < j_{\tilde{m}_1 - 1} < j_{\tilde{m}_1} = \frac{p_2}{\tilde{p}_1}$ be a sequence such that $j_{i+1} - j_i = r_2 + 1$ when $0 \le i < d/\tilde{p}_1$ and $j_{i+1} - j_i = r_2$ when $d/\tilde{p}_1 \le i \le \tilde{m}_1 \tilde{p}_1 - 1$. Define a p_2-periodic potential $f_2(l)$ for $0 \le l \le p_2 - 1$ as follows. Let j be such that $l \in I_j$ and let i be such that $j_{i-1} \le j < j_i$ and let $f_2(l) = \tilde{f}_i(l)$. For any sequence $\vec{t} = (t_1, t_2, \ldots, t_{\tilde{m}_1})$ with $t_i \in \{0, 1, \ldots, r_2 - 1\}$, let $f_2^{\vec{t}}$ be a p_2-periodic potential defined as follows. Let $0 \le l \le p_2 - 1$, and let j be such that $l \in I_j$. If $j = j_i - 1$ for some $1 \le i < \tilde{m}_1$, we let $f_2^{\vec{t}}(l) = f_2(l) + r_2^{-4} t_i$, and $j = j_{\tilde{m}_1} - 2$ then $f_2^{\vec{t}}(l) = f_2(l) + r_2^{-4} t_{\tilde{m}_1}$. Otherwise we let $f_2^{\vec{t}}(l) = f_2(l)$. Let p_2 be sufficiently large so that $p_2^{-2} < 1/3$.

Moreover, we can estimate the Lebesgue measure of the spectrum. For any $E \in \mathbb{R}$ and $\varepsilon_2 < |\lambda| < \frac{1}{\varepsilon_2}$, we can find $\tilde{f}_i \in \tilde{F}_1$ such that $L(E, \lambda \tilde{f}_i) > \tilde{\delta}_1$ since $L(E, \lambda \tilde{F}_1) > \tilde{\delta}_1$. If r_2 large enough, we have $\|A^{(E, \lambda \tilde{f}_i)}_{(r_2 - 2)\tilde{p}_1}\| > e^{\tilde{\delta}_1 (r_2 - 2)\tilde{p}_1}$. Then we have

$$\|A^{(E, \lambda f_2^{\vec{t}_k})}_{(r_2 - 2)\tilde{p}_1}(f_2^{\vec{t}_k}(j_{i-1}\tilde{p}_1))\| = \|A^{(E, \lambda \tilde{f}_i)}_{(r_2 - 2)\tilde{p}_1}\| > e^{\tilde{\delta}_1 (r_2 - 2)\tilde{p}_1}.$$

Since E is arbitrary, we can apply Lemma 2.11 to conclude that the total Lebesgue measure of $\Sigma(\lambda f_2^{\vec{t}_k})$ is at most $4\pi p_2 e^{-\tilde{\delta}_1 (r_2 - 2)\tilde{p}_1} < e^{-\tilde{p}_1 p_2^{1/2}}$ when r_2 sufficiently large. (Here $f_2^{\vec{t}_k}$ can be any element from F_2.)

Construction 3. Repeating the above procedures. Once we have constructed $F_{i-1} \subset B_{p_{i-1}^{-(i-1)}} \subset B_{\varepsilon_{i-1}}$, by Lemma 3.1, we can get a finite family of $\tilde{p}_{i-1}$-periodic potentials $\tilde{F}_{i-1} \subset B_{p_{i-1}^{-(i-1)}}$ satisfying the following. Let $\varepsilon_i = \frac{min\{\varepsilon_{i-1}, \delta_{i-1}\}}{10}$, and we have

$$L(E, \lambda \tilde{F}_{i-1}) > \tilde{\delta}_{i-1}$$

for some $0 < \tilde{\delta}_{i-1} < 1$ whenever $\forall \varepsilon_i < |\lambda| < \frac{1}{\varepsilon_i}$ and $E \in \mathbb{R}$, and

$$\left| L(E, \lambda \tilde{F}_{i-1}) - L(E, \lambda F_{i-1}) \right| < \frac{\varepsilon_i}{2}$$

whenever $|E| < \frac{1}{\varepsilon_i}$ and $|\lambda| < \frac{1}{\varepsilon_i}$.

Next, as in *Construction 2*, we will get a finite family F_i of p_i-periodic potentials which satisfies the following (here our perturbation is $r_i^{-Ni} t = r_i^{-2i} t$, $t \in \{0, 1, 2, \ldots, r_i - 1\}$).

(i). $L(E, \lambda F_i) > \delta_i$ for some $0 < \delta_i < 1$ and all $E \in \mathbb{R}$ and $\varepsilon_i < |\lambda| < \varepsilon_i^{-1}$.

(ii). $|L(E, \lambda F_i) - L(E, \lambda F_{i-1})| < \varepsilon_i$, for $|E| < \frac{1}{\varepsilon_i}$ and $|\lambda| < \frac{1}{\varepsilon_i}$.

(iii). $F_i \subset B_{p_i^{-i}} \subset B_{\varepsilon_i} \subset B_{\varepsilon_{i-1}} \subset B_{\varepsilon_0}(f), i > 2$. (Note: B_{ε_2} may not be in B_{ε_1}.)

(iv). $\forall f_i^{\vec{t}_k} \in F_i$, $|\Sigma(\lambda f_i^{\vec{t}_k})| \le e^{-\tilde{p}_{i-1} p_i^{1/2}}$ when $\varepsilon_i < |\lambda| < \varepsilon_i^{-1}$ (here $|\cdot|$ denotes the Lebesgue measure).

(v). $p_i^{-i} < \frac{1}{3}(i-1)^{-\tilde{p}_{i-1}}$ since we can let p_i be sufficiently large.

(vi). $\|f_i^{\vec{t}_1} - f_i^{\vec{t}_{m_i}}\| = \frac{4\pi}{\varepsilon_i \tilde{p}_{i-1} N_2(\tilde{p}_{i-1})} < \frac{1}{3}(i-1)^{-\tilde{p}_{i-1}}$. Here $N_2(\tilde{p}_{i-1})$ appears as in *Construction 1*, and we can ensure that this inequality holds since $\tilde{p}_{i-1}$ is fixed while $N_2(\tilde{p}_{i-1})$ can be taken

as large as needed.

Then we will get a limit-periodic potential $f_\infty \in B_{\varepsilon_0}(f)$, whose Lyapunov exponent is a positive continuous function of energy E and the Lebesgue measure of the spectrum is zero (Lemma 2.9 implies that $L(E, \lambda f_i^{\vec{t}}) \to L(E, \lambda f_\infty)$). Moreover, we have the following two claims.

Claim 3.5. *f_∞ is a Gordon potential.*

Proof. Let $q_i = \tilde{p}_i$. Obviously, $q_i \to \infty$ as $i \to \infty$. For $i \geq 1$, we have

$$\begin{aligned}\max_{1\leq n\leq q_i} |f_\infty(n) - f_\infty(n \pm q_i)| &\leq |f_\infty(n) - f_{i+1}^{\vec{t}_1}(n)| + |f_\infty(n \pm q_i) - f_{i+1}^{\vec{t}_1}(n \pm q_i)| \\ &\quad + |f_{i+1}^{\vec{t}_1}(n) - f_{i+1}^{\vec{t}_1}(n \pm q_i)| \\ &\leq p_{i+1}^{-(i+1)} + p_{i+1}^{-(i+1)} + \frac{4\pi}{\varepsilon_{i+1}\tilde{p}_i N_2(\tilde{p}_i)} \\ &\leq 2\frac{1}{3}(i)^{-\tilde{p}_i} + \frac{1}{3}(i)^{-\tilde{p}_i} \\ &\leq i^{-q_i}.\end{aligned}$$

So f_∞ is a Gordon potential. (Here $f_{i+1}^{\vec{t}_1}$ is an element of F_{i+1}). □

Claim 3.6. *$\Sigma(\lambda f_\infty)$ has zero Hausdorff dimension for every $\lambda \neq 0$.*

Proof. Let $\lambda \neq 0$ and $0 < \alpha \leq 1$ be given. Without loss of generality, assume $\lambda > 0$. Choose i large enough so that $\varepsilon_i < \lambda < 1/\varepsilon_i$ and $1/i < \alpha$. For every $f_i^{\vec{t}_k} \in F_i$, $\|\lambda f_\infty - \lambda f_i^{\vec{t}_k}\| < \lambda p_i^{-i}$ implies[2] $\mathrm{dist}(\Sigma(\lambda f_\infty), \Sigma(\lambda f_i^{\vec{t}_k})) < \lambda p_i^{-i}$. Since $\lambda f_i^{\vec{t}_k}$ is p_i-periodic, we have

$$\Sigma(\lambda f_i^{\vec{t}_k}) = \bigcup_{z=1}^{p_i} \tilde{I}_z^{(\vec{t}_k, i)},$$

where $\tilde{I}_z^{(\vec{t}_k,i)} = [a_z, b_z]$ is a closed interval.

Let $I_z^{(\vec{t}_k,i)} = [a_z - \lambda p_i^{-i}, b_z + \lambda p_i^{-i}]$ and since $\mathrm{dist}(\Sigma(\lambda f_\infty), \Sigma(\lambda f_i^{\vec{t}_k})) \leq \lambda p_i^{-i}$, we have

$$\Sigma(\lambda f_\infty) \subset \bigcup_{z=1}^{p_i} I_z^{(\vec{t}_k,i)}.$$

Moreover, $b_z - a_z \leq e^{-\tilde{p}_{i-1} p_i^{1/2}}$ since $|\Sigma(\lambda f_i^{\vec{t}_k})| \leq e^{-\tilde{p}_{i-1} p_i^{1/2}}$. Then we have

$$\begin{aligned}h^\alpha(\Sigma(\lambda f_\infty)) &\leq \lim_{i\to\infty} \sum_{z}^{p_i} (e^{-\tilde{p}_{i-1} p_i^{1/2}} + 2\lambda p_i^{-i})^\alpha \\ &= \lim_{i\to\infty} p_i (e^{-\tilde{p}_{i-1} p_i^{1/2}} + 2\lambda p_i^{-i})^\alpha \\ &= \lim_{i\to\infty} (p_i^{1/\alpha} e^{-\tilde{p}_{i-1} p_i^{1/2}} + 2\lambda p_i^{-i+1/\alpha})^\alpha.\end{aligned}$$

Since $1/i < \alpha$, we have $-i + 1/\alpha < 0$, and it follows that

$$\lim_{i\to\infty} (p_i^{1/\alpha} e^{-p_{i-1} p_i^{1/2}} + 2\lambda p_i^{-i+1/\alpha})^\alpha = 0.$$

[2]It is well known that for $V, W : \mathbb{Z} \to \mathbb{R}$ bounded, we have $\mathrm{dist}(\sigma(\Delta + V), \sigma(\Delta + W)) \leq \|V - W\|_\infty$, where $\mathrm{dist}(A, B)$ denotes the Hausdorff distance of two compact subsets A, B of $\mathbb{R}$.

So we have $h^\alpha(\Sigma(\lambda f_\infty)) = 0$ (note: when $i \to \infty$, λ belongs to $(\varepsilon_i, \frac{1}{\varepsilon_i})$ for all i large enough since this interval is expanding). So the Hausdorff dimension of the spectrum is less than α. Since α was arbitrary, the Hausdorff dimension must be zero. □

This implies all the assertions in Theorem 1.3 except for the absence of eigenvalues for every ω. Given the Gordon Lemma (see Lemma 3.4 above), this last statement will follow once Theorem 1.5 is established. □

Remark 3.7. *Since $\delta_i \le \delta_{i-1}/10, i \ge 1$, it is true that when $\varepsilon_i < |\lambda| < \frac{1}{\varepsilon_i}$, $L(E, \lambda f_\infty) \ge \frac{8}{9}\delta_i$ for any $E \in \mathbb{R}$. This gives information about the range of the Lyapunov exponent on certain intervals. Clearly, $\delta_i \to 0$ when $i \to \infty$ since the Lyapunov exponent will go to zero when λ goes to zero.*

Proof of Theorem 1.5. Let $\omega = e$ first. Relative to any minimal translation $\tilde{T}$, the selected f in the proof of Theorem 1.3 is still n_0-periodic by Proposition 2.7, so we can start with the same ball $B_{\varepsilon_0}(f)$ and choose the same periodic potentials in $B_{\varepsilon_0}(f)$. Then we get the same f_∞. For the finite family F_i from *Construction 3*, though the Lyapunov exponent may change, the following properties hold (note that $\|f_i^{\vec{t}_1}\|$ does not change).

(i). $F_i \subset B_{p_i^{-i}} \subset B_{\varepsilon_i} \subset B_{\varepsilon_0}(f)$.

(ii). $p_i^{-i} < \frac{1}{3}(i-1)^{-\tilde{p}_{i-1}}$.

(iii). $\|f_i^{\vec{t}_1} - f_i^{\vec{t}_{m_i}}\| = \frac{4\pi}{\varepsilon_i \tilde{p}_{i-1} N_2(\tilde{p}_{i-1})} < \frac{1}{3}(i-1)^{-\tilde{p}_{i-1}}$.

Then Claim 3.5 holds true, and so $f(\tilde{T}^n(e))$ is a Gordon potential. For arbitrary $\tilde{\omega}$, if we repeat the same procedures, (i)—(iii) above still hold as stated (since none of them are related to ω), and Theorem 1.5 follows. □

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